Introduction to Linear Time Invariant Systems

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Introduction

A system is some sort of black-box having one or more inputs and one or more outputs. Regarding the system as a black-box means that the interior of the system is out of reach or irrelevant. In the following only single input, single output systems are discussed. Examples of systems are digital filters, electrical networks and other physical phenomenons.
For a system to be linear the following is required:

1. Increasing the input by a factor k will increase the output by a factor of k.

2. The output of the sum of two inputs must be the sum of the outputs of the two inputs.

A system, where the input is a function of time, is regarded as time-invariant if a delay of the input just causes a delay of the output. The term time should be replaced by shift for digital filters.
Note that for such a system the shape of the input need not be identical to the shape of the output. An example is given in Figure 1. The change of shape is sometimes called linear distortion.
A broad class of physical phenomenons are approximately linear and time-invariant. For example a loudspeaker and microphone in a closed room, a hammer hitting a mechanical device or an electrical network. However, most systems behave non-linearly for large inputs.
An important quantity of a system is the impulse response. It is a rather strange quantity for a analog(physical) system, in that it is the output of the system when an infinitely short and infinitely high impulse with finite energy is applied. But for a digital system it is just the response to a single sample input with unit amplitude. If the digital system is known to be linear and shift invariant, it can be shown (see Figure 2) that the output can be written:

$$y(n) = \sum_{-\infty}^{\infty} h(n-k)x(k) = \sum_{-\infty}^{\infty} h(k)x(n-k)$$

where:
h is the impulse response of the system
x is the input to the system
For an analog(physical) system the corresponding relation is:

$$y(t) = \int_{-\infty}^{\infty} h(t-s)x(s)ds = \int_{-\infty}^{\infty} h(s)x(t-s)ds$$

This is called the convolution theorem and is often written:

$$y = h * x = x * h$$

It is important to note that the impulse response is all that is needed in order to calculate the output for any input. Put in another way: Knowing the impulse response, all is known about the system regarded as a black-box. Or alternatively: The impulse response fully characterizes the system. This is why the impulse response is such an important quantity.
If theoretically a complex exponential is used as input, the output can in the digital case be written:

$$y(n) = \sum_{-\infty}^{\infty} h(k)e^{j\omega(n-k)} = \sum_{-\infty}^{\infty} h(k)e^{-j\omega k}e^{j\omega n} = H(\omega)e^{j\omega n}$$

where

$$H(\omega) = \sum_{-\infty}^{\infty} h(k) e^{-j\omega k}$$

The term H is called the system frequency response. Since h can be found from H:

$$h(n) = \frac{1}{2\pi} \int_0^{2\pi} H(\omega) e^{j\omega n} d\omega$$

H also fully characterizes a linear and time(shift) invariant system.
Very often H can be measured with a greater accuracy than h and also gives a more desirable view of the system.
Note that H is independent of n. This means that the output for a sine wave input is also a sine wave with the same frequency but with a different amplitude and phase in opposition to the general case where the shape changes from input to output as mentioned above.
If a periodic signal with period N is applied then the output will also be periodic. Moreover if h has a length shorter than N, then the convolution for one period becomes:

$$y(n) = \sum_{0}^{N-1} h(k) x(n-k)$$

This is also called the circular convolution of the finite sequences h and x.
Theoretically this set of linear equations could be solved for h, but a method requiring much less processing power is using the Fast Fourier Transform. Multiplying by a complex exponential and summing gives:

$$\begin{aligned}
\sum_{0}^{N-1} y(n) e^{-j\frac{2\pi}{N}mn} &= \sum_{0}^{N-1} \sum_{0}^{N-1} h(k) x(n-k) e^{-j\frac{2\pi}{N}mn} \\
&= \sum_{0}^{N-1} h(k) \sum_{0}^{N-1} x(p) e^{-j\frac{2\pi}{N}m(p+k)} \\
&= \sum_{0}^{N-1} h(k) e^{-j\frac{2\pi}{N}mk} \sum_{0}^{N-1} x(p) e^{-j\frac{2\pi}{N}mp}
\end{aligned}$$

or

$$Y(m) = H(m) X(m)$$

Here each term is the Discrete Fourier Transform of y, h and x. It is seen that the circular convolution transforms into a product making it possible to find H from x and y using FFT.
If h and y are known, X and thereby x can be calculated. This process is called deconvolution.

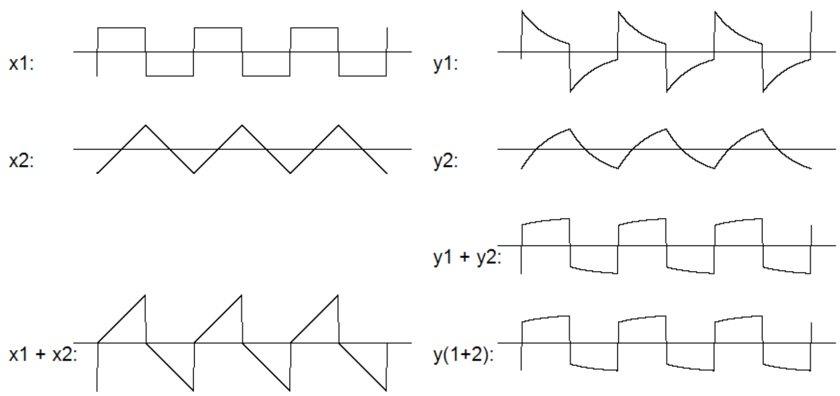
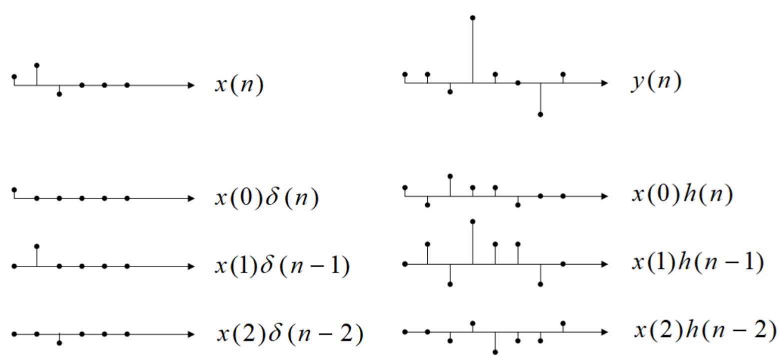


Figure 1: Linear system



$$
\begin{aligned}
y(n) &= \sum_k x(k)h(n-k) \\
&= \sum_k h(k)x(n-k)
\end{aligned}
$$

Figure 2: Convolution