

# Interpolation for sample rate change

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# Change log

## **10. June 2019**

1. Document started.

## **7. March 2020**

1. Meta data added.

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# Chapter 1

## Introduction

Interpolation is used in many situation, but in the following the focus is on signal sample rate change. Here the sampling frequency is relatively high and the resampling is often done in real time. If the input and output sampling frequencies are fixed and the ratio is a rational number with a relatively small numerator, then a FIR filter can be designed to do the resampling. Several techniques exist for the design of such a filter, e.g. Remez exchange algorithm and windowing design. These techniques are not covered here.

In some situations the sampling ratio is varying with time and it is no longer possible to use a fixed filter. In other situations the sampling ratio is a rational number with a large numerator making the design of the filter impractical.

In the next two chapters two techniques are reviewed: Lagrange interpolation and Uwe Hermann interpolation. For both methods the interpolated values are calculated as a weighted sum of a set of the given data and can thus be viewed as a FIR filtering process since the weights are the same for all intervals:

$$f(s) \approx \sum_p h(s-p)f(p) \quad (1.1)$$

Letting  $s = n + t$  one has:

$$f(n+t) \approx \sum_p h(n-p+t)f(p), \quad 0 \leq t < 1 \quad (1.2)$$

$$= \sum_{m=-L+1}^L h(-m+t)f(n+m) \quad (1.3)$$

$$= \sum_{m=-L+1}^L w_m(t)f(n+m) \quad (1.4)$$

and consequently:

$$h(-m+t) = w_m(t) \quad (1.5)$$

# Chapter 2

## Lagrange interpolation

Lagrange interpolation fits a polynomial to  $N$  consecutive data values. Usually  $N = 2L$  and the polynomial is thus of odd order  $2L - 1$ . It is now possible to interpolate by sampling the middle interval of the polynomial. A new polynomial is calculated for each interval of interest, see Figure 2.1.

When the input data samples are equally spaced with a spacing of 1, the interpolation is given by:

$$f(n+t) \approx \sum_{m=-L+1}^L w_m(t) f(n+m), \quad 0 \leq t < 1, \quad L \geq 1 \quad (2.1)$$

$$w_m(t) = \prod_{\substack{l=-L+1 \\ l \neq m}}^L \frac{t-l}{m-l} \quad (2.2)$$

or in barycentric form:

$$f(n+t) \approx \frac{\sum_{m=-L+1}^L \frac{P_m}{t-m} f(n+m)}{\sum_{m=-L+1}^L \frac{P_m}{t-m}} \quad (2.3)$$

$$P_m = \frac{1}{\prod_{\substack{l=-L+1 \\ l \neq m}}^L (m-l)} \quad (2.4)$$

The set  $P_m$  is independent of  $t$  and need only be calculated once. I.e. for each value of  $t$  only  $O(L)$  operations are required. If the order is relatively low, it is possible to do the interpolation in real time.

The filter impulse response for  $L = 1, 3, 10$  is shown in Figure 2.2. Note that  $L = 1$  corresponds to linear interpolation.

The corresponding spectra are shown in Figure 2.3.

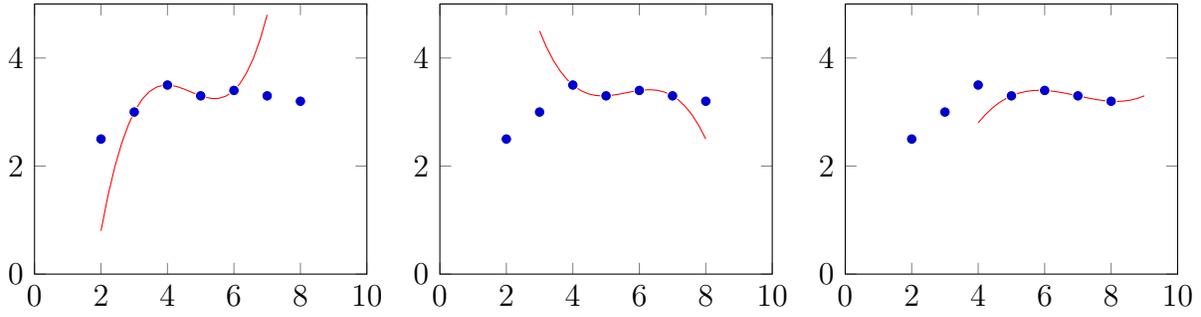


Figure 2.1: Lagrange interpolation for  $2L = 4$ . Only the center interval is used:  $n < n+t < n+1$ . Left graph  $n = 4$ , middle graph  $n = 5$ , right graph  $n = 6$

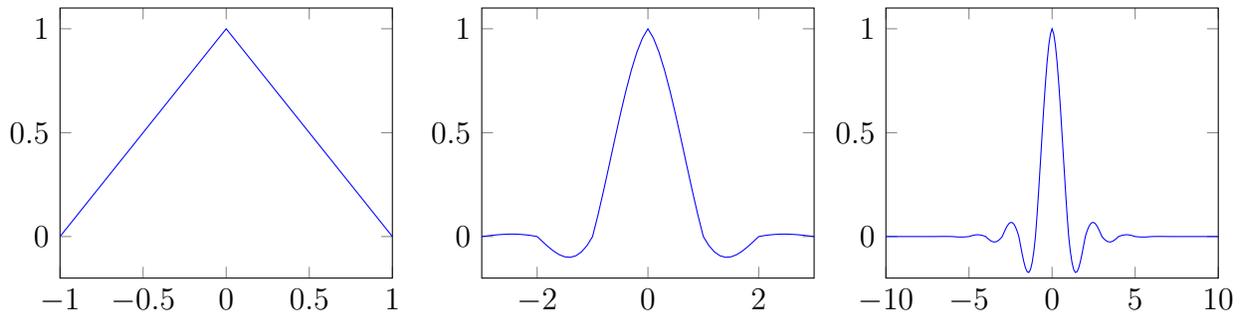


Figure 2.2: Impulse response of Lagrange interpolation for  $L = 1, 3, 10$

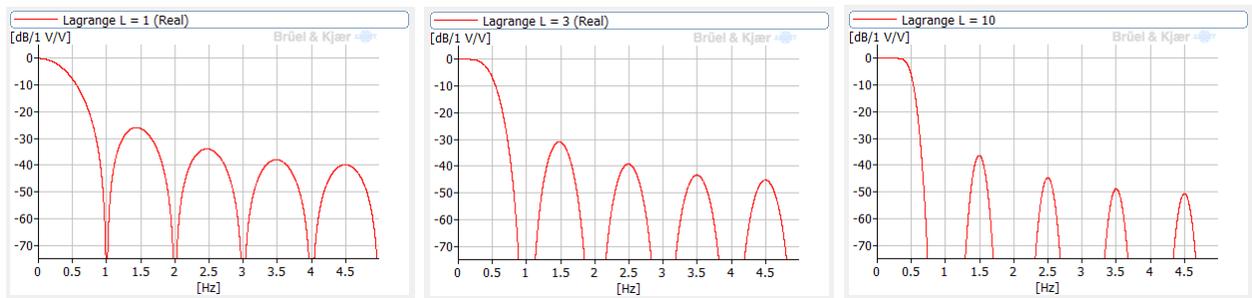


Figure 2.3: Spectrum of the impulse response of Lagrange interpolation for  $L = 1, 3, 10$

# Chapter 3

## Uwe Hermann interpolation

Uwe Hermann interpolation [1] also uses a weighted sum of neighbouring values, but here the sum of the squared interpolation errors of all complex exponential within a given frequency interval is minimized.

$$f(n+t) \approx \sum_{m=-L+1}^L w_m(t) f(n+m), \quad 0 \leq t < 1, \quad L \geq 1 \quad (3.1)$$

$$e^{j\omega(n+t)} \approx \sum_{m=-L+1}^L w_m(t) e^{j\omega(n+m)} \quad (3.2)$$

The error sum is given by:

$$E(w, t) = \int_{-\omega_c}^{\omega_c} |e^{j\omega(n+t)} - \sum_{m=-L+1}^L w_m(t) e^{j\omega(n+m)}|^2 d\omega \quad (3.3)$$

and the partial derivative with respect to  $w_m$  is given by:

$$\frac{\partial E(w, t)}{\partial w_k} = 2 \int_{-\omega_c}^{\omega_c} \text{Re}((e^{-j\omega(n+t)} - \sum_{m=-L+1}^L w_m(t) e^{-j\omega(n+m)}) e^{j\omega(n+k)}) d\omega \quad (3.4)$$

$$= -2 \int_{-\omega_c}^{\omega_c} \text{Re}(e^{-j\omega(t-k)} - \sum_{m=-L+1}^L w_m(t) e^{-j\omega(m-k)}) d\omega \quad (3.5)$$

Setting the partial derivative equal to zero:

$$\int_{-\omega_c}^{\omega_c} (\cos \omega(t-k) - \sum_{m=-L+1}^L w_m(t) \cos \omega(m-k)) d\omega = 0 \quad (3.6)$$

$w_m$  can be found by solving the  $2L$  linear equations:

$$\sum_{m=-L+1}^L w_m(t) \frac{\sin \omega_c(m-k)}{(m-k)} = \frac{\sin \omega_c(t-k)}{(t-k)} \quad (3.7)$$

Since the matrix on the left hand side is independent of  $t$ , the matrix need only be inverted once and the calculation of  $w_m$  requires  $O(L^2)$  operations.

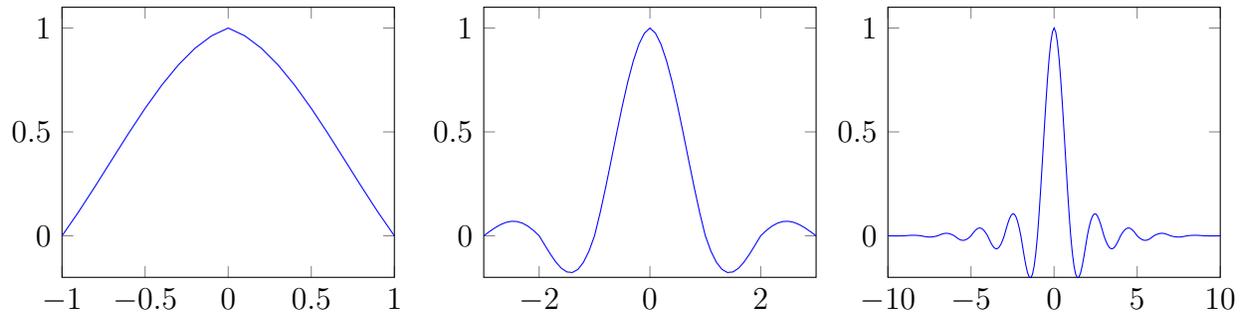


Figure 3.1: Impulse response of Uwe Hermann interpolation for  $L = 1, 3, 10$ .  $\omega_c = 0.4 * 2\pi$

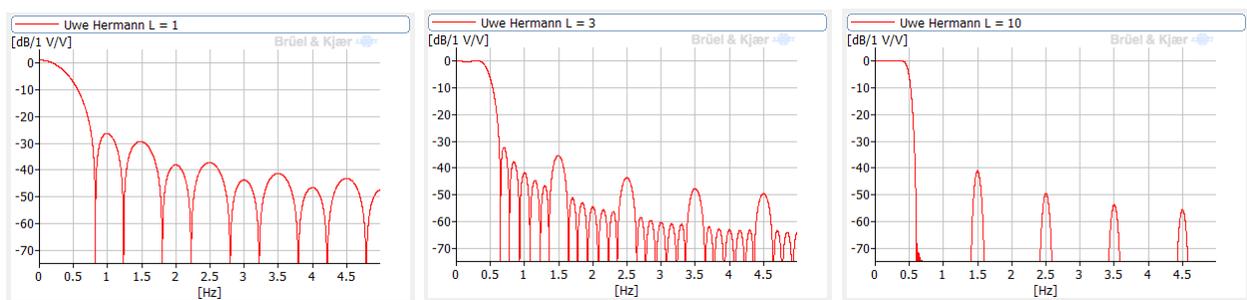


Figure 3.2: Spectrum of the impulse response of Uwe Hermann interpolation for  $L = 1, 3, 10$

# Chapter 4

## Comparison between Lagrange and Uwe Hermann interpolation

The order  $L$  is typically considerably lower for Uwe Hermann interpolation than for Lagrange interpolation as can be seen from Figure 4.1. The figure also shows a FIR filter designed using Remez exchange algorithm. The number of weights required for the interpolation becomes 100, 20, and 17 for Lagrange, Uwe Hermann and Remes. Asuming an interpolation factor of 10 for the later.

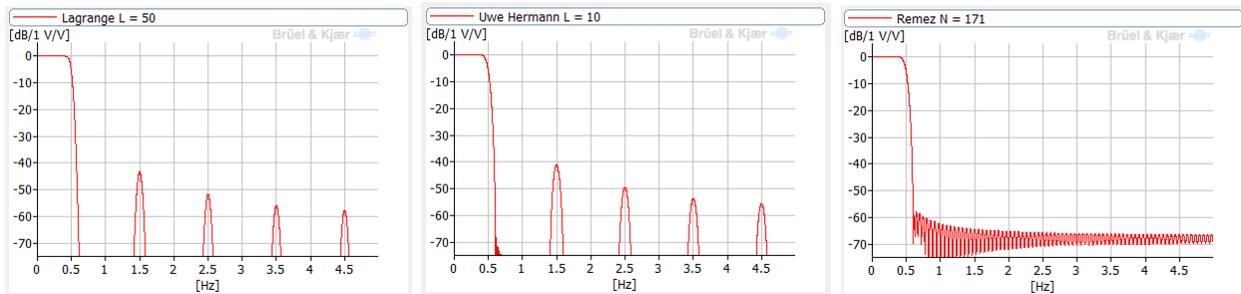


Figure 4.1: Spectrum of impulse response

# Bibliography

- [1] Diplomarbeit für Uwe Hermann, "Entwurf interpolierender digitaler Systeme zur weitgehend beliebigen Änderung der Taktfrequenzen", Erlangen 1984