Introduction to Gödel theory

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1. "Truth that cannot be proven" clarified

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1. "Truth that cannot be proven" changed once more

Introduction

In 1931 Kurt Gödel published his two inconsistency Theorems. Unfortunately it is often unclear what Gödel proved and wrote about. I shall in the following try to clarify some of the consequences of his theories.

Sentence numbers

Any sentence may correspond to a unique integer (sentence number) and any integer may correspond to a unique sentence. This can be done by allocating an integer to every symbol in the sentences of interest. As an example, the integers 1-52 may correspond to the letters A - z, if these are the only symbols in the sentences. In general sentences also contain other symbols.
The sentence number is now calculated by the formula:

$$s = 2^{n_1}3^{n_2}5^{n_3}7^{n_4} \;.\;.\;.\;.\; p_l^{n_l}$$

where n_i is the integer for the i'th symbol in the sentence and p_i is the i'th prime number. The number of symbols in the sentence is l.
See D. Hofstader: Limits of Logic: The Gödel Legacy
There is thus an isomorphism between the integers > 1 and the set of sentences, since the prime factoring is unique.

Well formed sentenses

If a sentence is well formed (without self reference) and can be proven true, it is called a theorem. The sentence number is then called a theorem number.
An example of a self referencing sentence is:

This sentence is false

Before Gödel published his theories it was assumed that all well formed sentences could be proven either true or false.

An example

The sentence:

H = {The number h is not a prime number}

has a sentence number that is easily computed. H is a well formed sentence, it is not self referencing. The number can be checked and H may therefore be proven true and consequently a theorem or proven false.

Definition of the number g

The sentence:

G = {The number g is not a theorem number}

has a sentence number that is easily computed. This number is called g. In the following g refers to this specific number.

Here is the beef

We now turn to what it's all about, the sentence:

G = {The number g is not a theorem number}
G' = {The number g is a theorem number}

where g refers to the number above.
Can we prove G is true and thereby a theorem? And how do we do it?
Can we prove G is false. And how do we do it?
For a sentence S to be proven true, its negation S' must be proven false.
For a sentence S to be proven false, just assume S is true and see if it leads to a contradiction.

Prove G is true is the same as prove G' false:
Assume G' true:
No contradiction. We could therefore not prove G' is false. And we could not prove G is true.

Prove G false:
Assume G true:
No contradiction. We could therefore not prove G is false.

It is therefore not possible to say whether the sentence G is true or false. One can say that Gödel has smuggeled self reference in through a backdoor like a trojan horse by making an isomorphism between sentences and integers. The sentence G only expresses something about a number and is not self referencing, but as the number g is a sentence number, we are in a way dealing with self referencing.

Truth that cannot be proven

One of the results of Gödels theory is often stated as ”There are true sentences, that cannot be proven”. This may sound strange from a formal point of view, but if mathematics is regarded as something we do not invent but discover, then it may have a meaning, but this is more of a philosophical discussion.